Announcements

1) Proof to contraction mapping theorem fixed online

2) Problem day : Monday

Remark: In the statement of the inverse function theorem, "f' continuous" means as a map from E into the linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Inverse Function Theorem

Let $f : E \to \mathbb{R}^n$ where $E \subseteq \mathbb{R}^n$ is open. Suppose $\exists$ an $a \in E$ with $f'(a)$ invertible. If f is differentiable on E, and f' is continuous on E

a) $\exists$ $\varepsilon > 0$ such that

f is 1-1 on $B(a,\varepsilon) = U$,

and moreover, $f(U)$ is open.

b) If we let $g = f^{-1} : f(U) \to U$,

then g is differentiable

on $f(U)$, and

$$g'(y) = \left(f'(f^{-1}(y))\right)^{-1}$$

$\forall$ $y \in f(U)$.

<u>proof:</u> a) Since f' is continuous, $\exists\ \delta > 0$ such that 1) $B(a,\delta) \subseteq E$

2) $\|f'(a) - f'(x)\| < \dfrac{1}{2\|(f'(a))^{-1}\|}$

Let $A = f'(a)$. We then know that $f'(x)$ is invertible $\forall\ x \in B(a,\delta)$. Let

$$\varepsilon = \frac{1}{2\|A^{-1}\|}.$$

Trick: Let $y \in \mathbb{R}^n$. Define

$$\boxed{\varphi_y(x) = x + A^{-1}(f(y) - f(x))}.$$

Suppose $f(x) = f(y)$.

Then $\varphi_y(x) = x$, so x will be a fixed point for φ_y. But we know $\varphi_y(y) = y$ by definition, so if we can show φ_y has a unique fixed point, we get f is injective.

Let $x \in B(a, \delta)$ and

consider

$$\varphi_y'(x) = I_n - A^{-1} f'(x),$$

hence

$$\begin{aligned}
\|\varphi_y'(x)\| &= \|I_n - A^{-1} f'(x)\| \\
&= \|A^{-1}A - A^{-1} f'(x)\| \\
&\leq \|A^{-1}\| \, \|A - f'(x)\| \\
&< \|A^{-1}\| \frac{1}{2\|A^{-1}\|} \\
&= \frac{1}{2}
\end{aligned}$$

We then know that φ_y' is uniformly bounded by $1/2$ on $B(a,\delta)$, so by our theorem finished in last class,

$$\|\varphi_y(x_1)-\varphi_y(x_2)\|_2 \leq \frac{1}{2}\|x_1-x_2\|_2$$

$\forall\ x_1, x_2 \in B(a,\delta)$.

(recall $B(a,\delta)$ is convex)

By the contraction mapping theorem, φ_y has a unique fixed point, which must be y.

Therefore, f is injective on $B(a, \delta) = U$.

Now we want to show that $f(U) = V$ is open. Let $y_0 \in V$.

Show $\exists$ $\alpha > 0$ with $B(y_0, \alpha) \subseteq V$.

Since f is injective on U, there is a unique $x_0 \in U$, $y_0 = f(x_0)$.

Choose $\gamma > 0$ so that

$\overline{B(x_0, \gamma)} \subseteq U$.

Consider, for $x \in \overline{B(x_0, \gamma)}$

$$\| \varphi_y(x) - x_0 \|_2$$

$$= \| \varphi_y(x) - \varphi_y(x_0) + \varphi_y(x_0) - x_0 \|_2$$

$$\leq \| \varphi_y(x) - \varphi_y(x_0) \|_2$$

$$+ \| \varphi_y(x_0) - x_0 \|_2$$

Now

$$\| \varphi_y(x_0) - x_0 \|_2$$

$$= \| x_0 + A^{-1}(y - f(x_0)) - x_0 \|_2$$

$$= \| A^{-1} (y - f(x_0)) \|_2$$

$$\leq \| A^{-1} \| \, \| y - f(x_0) \|_2$$

$$= \| A^{-1} \| \, \| y - y_0 \|_2$$

If $\quad \| y - y_0 \|_2 < \dfrac{\gamma}{2 \| A^{-1} \|}$,

we get

$$\| \varphi_y(x_0) - x_0 \|_2 < \frac{\gamma}{2}.$$

Also,

$$\|\varphi_y(x) - \varphi_y(x_0)\|_2$$

$$\leq \frac{1}{2} \|x - x_0\|_2$$

Since $x, x_0 \in \overline{B(x_0, r)} \subseteq U$.

Then we get

$$\|\varphi_y(x) - \varphi_y(x_0)\|_2$$

$$\leq \frac{1}{2} r$$

So if $\alpha = \dfrac{\gamma}{2\|A^{-1}\|}$

and $y \in B(y_0, \alpha)$, then

$\|\varphi_y(x) - x_0\|$

$< \dfrac{\gamma}{2} + \dfrac{\gamma}{2} = \gamma.$

$\Rightarrow \varphi_y\left(\overline{B(x_0, \gamma)}\right) \subseteq \overline{B(x_0, \gamma)}.$

Then by the contraction mapping theorem,

φ_y has a unique fixed point in $\overline{B(x_0, \delta)}$.

If x denotes this fixed point,

$$x = \varphi_y(x) = x + A^{-1}(y - f(x))$$

$$\Longrightarrow \qquad y = f(x)$$

This implies

$$y \in f\left(\overline{B(x_0, r)}\right) \subset f(U) = V$$

$$\Longrightarrow B(y_0, \alpha) \subseteq V$$

$$\Rightarrow V \text{ open.}$$

b) We want to show $g = f^{-1}$ is differentiable on V. Choose $y \in V$. Then $\exists$ a unique $x \in U$, $y = f(x)$.

Moreover, if we consider $y+k \in V$, $\exists$ a unique h with $x+h \in U$, $f(x+h) = y+k$.

Let $T = (f'(g(y)))^{-1}$

$$\| g(y+k) - g(y) - Tk \|_2$$

$$= \| g(f(x+h)) - g(f(x)) - Tk \|_2$$

$(g = f^{-1})$

$$= \| x + h - x - Tk \|_2$$

$$= \| h - Tk \|_2$$

$$= \| h - T(f(x+h) - f(x)) \|_2$$

$$= \| TT^{-1}h - T(f(x+h) - f(x)) \|_2$$

$$\leq \|T\| \; \| T^{-1}h - f(x+h) + f(x) \|_2$$

We wanted

$$\lim_{k \to 0} \frac{\| g(y+k) - g(y) - Tk \|_2}{\| k \|_2} = 0$$

We want to bound

$\frac{1}{\|k\|_2}$ by some constant

times $\frac{1}{\|h\|_2}$.

Now if $x+h, x \in U$,

$$\begin{aligned}
&\|h - A^{-1}k\|_2 \\
&= \|h + A^{-1}(y - (y+k))\|_2 \\
&= \|h + A^{-1}(f(x) - f(x+h))\|_2 \\
&= \|\varphi_y(x+h) - \varphi_y(x)\|_2 \\
&\leq \frac{1}{2}\|x+h-x\|_2 = \frac{1}{2}\|h\|_2
\end{aligned}$$

This implies

$$\begin{aligned}
\|h\|_2 &\leq \frac{1}{2}\|h\|_2 + \|A^{-1}k\|_2 \\
&\leq \frac{1}{2}\|h\|_2 + \|A^{-1}\| \cdot \|k\|_2
\end{aligned}$$

Hence,

$$\frac{\|h\|_2}{2} \leq \|A^{-1}\| \, \|k\|_2$$

$$\Rightarrow \quad \frac{1}{\|k\|_2} \leq \frac{2\|A^{-1}\|}{\|h\|_2}$$

We then know

$$\frac{\|g(x+k)-g(x)-Tk\|_2}{\|k\|_2}$$

$$\leq \|T\| \frac{\|f(x+h)-f(x)-T^{-1}k\|_2}{\|k\|_2}$$

$$\leq 2\|T\| \cdot \|A^{-1}\| \frac{\|f(x+h)-f(x)-T^{-1}k\|_2}{\|h\|_2}$$

$\longrightarrow 0$ as $k \to 0$ since

$k \to 0 \Rightarrow h \to 0$ and

f is differentiable at x with derivative T^{-1}. $\square$

Remark: In fact, under the hypotheses of the Inverse Function Theorem, g' is a continuous map from $V = f(U)$ to the linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$.